

Selection Examination for the Training Program

Mathematics

- There are FOUR questions in total.
- Be sure to attempt the ALL questions.
- Examinees have 3 hours.
- No examinee is allowed to leave the examination room during the first 1 hour and the last 10 minutes.
- Books, note books, dictionaries, mobile phones, and information devices must be kept in the specified place.

[Important Notes]

- 1: Do not turn this page until instructed to do so by the proctor.
- 2: There are 6 pages including this cover page. Please make sure you are not missing a page after the exam begins.
- 3: Be sure to write your name and examinee's number on all the answer sheets.
- 4: Use one answer sheet per question. Be sure to write the question number on the sheet.
- 5: Write answers in English or Japanese.
- 6: You may use the reverse side of answer sheet as calculation sheet.
- 7: If you need to use two or more sheets for one problem, be sure to mention at the end of the answer sheet that the answer continues on the next sheet.
- 8: When you need an additional answer sheet, notify the proctor by raising your hand, and receive a new answer sheet.
- 9: After the exam finishes, sort the answer sheets numerically, by question number, and fold in half.
- 10: You may take this booklet with you after the exam finishes.

[Notations]

Let \mathbb{N} , \mathbb{Z} , \mathbb{Q} , \mathbb{R} , and \mathbb{C} be the set of positive integers, integers, rational numbers, real numbers, and complex numbers, respectively.

THIS PAGE CONTAINS NO QUESTIONS.

1 Let $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ be the functions from \mathbb{R}^2 to \mathbb{R} defined by

$$\begin{aligned}\|x\|_1 &= |x_1| + |x_2|, \\ \|x\|_2 &= \sqrt{x_1^2 + x_2^2}, \\ \|x\|_\infty &= \max\{|x_1|, |x_2|\},\end{aligned}$$

for $x = (x_1, x_2)^T \in \mathbb{R}^2$, where \cdot^T denotes the matrix transpose operation.

Answer the following questions.

- (i) Sketch each of the sets $\{x \in \mathbb{R}^2 ; \|x\|_1 = 1\}$, $\{x \in \mathbb{R}^2 ; \|x\|_2 = 1\}$, and $\{x \in \mathbb{R}^2 ; \|x\|_\infty = 1\}$.
- (ii) Find positive constants C_1 , C_2 , and C_3 such that

$$\|x\|_1 \leq C_1 \|x\|_2, \quad \|x\|_2 \leq C_2 \|x\|_\infty, \quad \|x\|_\infty \leq C_3 \|x\|_1, \quad \forall x \in \mathbb{R}^2.$$

- (iii) Let $x \in \mathbb{R}^2$ be a point, and let $y_n = (y_{n,1}, y_{n,2})^T \in \mathbb{R}^2$ be a sequence such that

$$\lim_{n \rightarrow \infty} \|x - y_n\|_2 = 0.$$

Calculate $\lim_{n \rightarrow \infty} \|x - y_n\|_1$ and $\lim_{n \rightarrow \infty} \|x - y_n\|_\infty$.

2 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by

$$f(t) = \int_0^1 (\sin(\pi x) - tx(1-x))^2 dx.$$

Find a minimizer t of f . You may use the fact that $\int_0^1 x^2 \sin(\pi x) dx = \frac{1}{\pi} - \frac{4}{\pi^3}$ without verifying it.

3 Give an example to each of the following items.

- (i) A pair of non-zero 2-by-2 matrices, A and B , such that $AB \neq BA$.
- (ii) A pair of a non-zero 2-by-2 matrix A and a positive integer n such that $A^n = 0$.
- (iii) A 2-by-2 matrix A such that $x^T Ax > 0$ holds for all non-zero vector $x \in \mathbb{R}^2$.

4 Let D be the set of \mathbb{R}^2 defined by

$$D = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 ; \frac{7}{16}x_1^2 + \frac{3\sqrt{3}}{8}x_1x_2 + \frac{13}{16}x_2^2 < 1 \right\}.$$

Answer the following questions.

- (i) Find a symmetric matrix A such that $x^T Ax = \frac{7}{16}x_1^2 + \frac{3\sqrt{3}}{8}x_1x_2 + \frac{13}{16}x_2^2$ holds for all $x = (x_1, x_2)^T \in \mathbb{R}^2$.
- (ii) Let $R = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$. Calculate RAR^{-1} , where R^{-1} is the inverse matrix of R .
- (iii) Calculate $\iint_D dx_1 dx_2$.

THIS PAGE CONTAINS NO QUESTIONS.

プログラム履修者選抜試験問題

数学

- この問題冊子には問題が 4 問 ある。
- すべての問題 を解答せよ。
- 解答時間は 3 時間 である。
- 試験開始後 1 時間から 2 時間 50 分の間 は途中退席してもよい。
- 参考書・ノート類・辞書・携帯電話・情報機器等は、指定された荷物置場に置くこと。

[注意]

- 1: 指示のあるまで問題冊子を開かないこと。
- 2: この問題冊子は 6 ページ ある。試験開始後に全てのページがあるか確認すること。
- 3: 解答用紙のすべてに氏名・受験番号を記入すること。
- 4: 解答用紙は問題ごとに別の用紙を用い、それぞれ解答した問題番号を記入すること。
- 5: 解答は日本語または英語のどちらでもよい。
- 6: 解答用紙の裏面は計算用紙として使ってよい。
- 7: 一問を二枚以上にわたって解答するときは、つづきのあることを用紙下端に明示して次の用紙に移ること。
- 8: 解答用紙が足りなくなったら挙手により試験監督を呼び、新しい解答用紙を受け取ること。
- 9: 提出の際は解答用紙を問題番号順に重ね、記入した面を外にし、一括して二つ折りにして提出すること。
- 10: この問題冊子は持ち帰ってもよい。

[記号について]

設問中の N, Z, Q, R, C はそれぞれ正の整数, 整数, 有理数, 実数, 複素数の集合を表す。