Selection Examination for the Training Program
[for the 4-year Doctor's Course]

Mathematics

1 (THIS QUESTION MUST BE ANSWERED)

Answer the following questions.

(i) Find three numbers A, B, C satisfying

$$\frac{1}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2} \quad \text{for all } x \in \mathbb{R}.$$

(ii) Compute

$$\int_1^\infty \frac{1}{x(x+1)(x+2)}\, dx.$$

(iii) Compute

$$\sum_{n=1}^\infty \frac{1}{n(n+1)(n+2)}.$$

$\boxed{\mathbf{2}}$ (THIS QUESTION MUST BE ANSWERED)

Answer the following questions.

(i) Find a 2-by-2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying

$$A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad \text{and} \quad A \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}.$$

(ii) For $x \in \mathbb{R}$, let $B(x)$ be the 3-by-3 matrix

$$B(x) = \begin{pmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{pmatrix}.$$

Compute $\det B(x)$.

(iii) Let $B(x)$ be as in (ii). Sketch the graph of the function $f(x) = \operatorname{rank} B(x)$.

3 (Selection problem)

Let A be the 3-by-2 matrix, and b and c be the 3-dimensional vectors given by

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad c = \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}.$$

Let x and y denote 2-dimensional vectors. Answer the following questions.

(i) Solve the simultaneous linear equations $Ax = b$ if there exists a solution. Do the same for $Ay = c$.

(ii) Solve the simultaneous linear equations $A^T Ax = A^T b$, where A^T is the transpose of the matrix A.

(iii) Let x_0 be the solution to $A^T Ax = A^T b$ that you found in (ii). Prove that

$$\|A(x_0 + y) - b\| \geq \|Ax_0 - b\|, \quad \forall y \in \mathbb{R}^2,$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^3$.

4 (Selection problem)

Consider the following simultaneous ordinary differential equations:

$$\begin{cases} \dfrac{du_1}{dt}(t) = -au_1(t) + bu_2(t), \quad \forall t > 0, & \text{(1a)} \\ \dfrac{du_2}{dt}(t) = au_1(t) - bu_2(t), \quad \forall t > 0, & \text{(1b)} \\ u_1(0) = c \in \mathbb{R}, & \text{(1c)} \\ u_2(0) = d \in \mathbb{R}, & \text{(1d)} \end{cases}$$

where a and b are positive constants. Answer the following questions.

(i) Find a conserved quantity for equations (1a)–(1d).

(ii) Find the solution $u_1(t)$.

(iii) Let $c = 0$. Then, for a positive constant M, find a condition on d that is necessary and sufficient for $u_1(t)$ to satisfy $u_1(t) \leq M$ for all $t \geq 0$.

$\boxed{5}$ (Selection problem)

For the function $f(x, y) = \log(x^2 + y^2)$, answer the following questions.

(i) Find the equation of the tangent plane to the surface $z = f(x, y)$ at the point

$$(x_0,\ y_0,\ z_0) = \left(\frac{\sqrt{3}}{2},\ \frac{1}{2},\ 0\right).$$

(ii) Compute

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

(iii) Let $D(a) = \{(x, y) \mid x^2 + y^2 \leq a^2\}$ be the disc of radius $a > 0$ centered at the origin and let $g(a)$ be the function

$$g(a) = \iint_{D(a)} f(x, y)\, dxdy.$$

Find the minimum of $g(a)$ $(a > 0)$.

$\boxed{6}$ (Selection problem)

Let $A(1, 1, 0)$, $B(0, 2, 1)$, $C(1, 0, 1)$, and $D(2, 1, 1)$ be four points given in the xyz-space. Answer the following questions.

Remark. The phrase “line AB” means the straight line through the points A and B extended infinitely, *not* the line segment with endpoints A and B. A similar remark applies to “the plane through A, B, and C.”

(i) Find the minimum distance from the point C to the line AB.

(ii) Find the minimum distance from the point D to the points on the plane through A, B, and C.

(iii) Find the minimum distance from the points on the line AB to the points on the line CD.