Selection Examination for the Training Program

Mathematics

1 (THIS QUESTION MUST BE ANSWERED)

Let $f(x) = \sin(x)$. For an arbitrary non-negative integer N, let f_N and R_N be defined by

$$f_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n, \quad R_N(x) = f(x) - f_N(x), \quad \forall x \in \mathbb{R},$$

where $f^{(n)}$ is the n-th order derivative of f. Note that for each $x \in [0, 1]$ there exists $c \in (0, x)$ such that $R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1}$ by Taylor's theorem. Answer the following questions.

(i) Calculate $\displaystyle\int_0^1 \frac{f_6(x)}{x}\, dx$.

(ii) Find an upper bound of $\displaystyle\left|\int_0^1 \frac{R_6(x)}{x}\, dx\right|$.

(iii) The sinc function is defined by $\mathrm{sinc}(x) = \frac{\sin(x)}{x}$. Find an approximation for $\displaystyle\int_0^1 \mathrm{sinc}(x)\, dx$ and then estimate the absolute error between the exact value and your approximation.

$\boxed{\mathbf{2}}$ (THIS QUESTION MUST BE ANSWERED)

Let A be a 3-by-3 matrix defined by

$$A = \begin{pmatrix} 0 & -2 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Answer the following questions.

(i) Find all eigenvalues of A.

(ii) Find a 3-by-3 matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

(iii) Let e^A be defined by $e^A = \sum_{n=0}^{\infty} \dfrac{A^n}{n!}$. Calculate each elements of e^A.

3 (Selection problem)

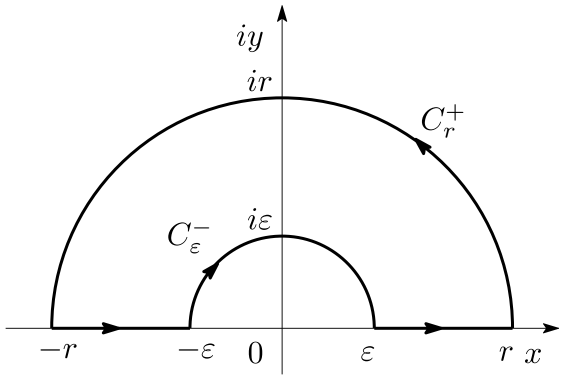
Euler's formula states that, for an arbitrary real number θ,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

holds, where e is the base of the natural logarithm, i is the imaginary unit.

For arbitrary positive numbers r and ε satisfying $\varepsilon < r$, let C_r^+ and C_ε^- be semicircles oriented as indicated in the figure on the right.

Answer the following questions.



(i) Calculate $\displaystyle\lim_{r\to\infty}\int_{C_r^+}\frac{e^{iz}}{iz}\,dz$ and $\displaystyle\lim_{\varepsilon\to 0}\int_{C_\varepsilon^-}\frac{e^{iz}}{iz}\,dz$.

(ii) Prove $\displaystyle\int_\varepsilon^r \frac{\sin(x)}{x}\,dx = \frac{1}{2}\left(\int_{-r}^{-\varepsilon}\frac{e^{ix}}{ix}\,dx + \int_\varepsilon^r \frac{e^{ix}}{ix}\,dx\right)$.

(iii) Calculate $\displaystyle\int_0^\infty \frac{\sin(x)}{x}\,dx$.

4 (Selection problem)

Let λ be a positive constant. We consider the following ordinary differential equation:

$$x\frac{d^2u}{dx^2}(x) + 2\frac{du}{dx}(x) + \lambda^2 x u(x) = 0, \quad \forall x \in \mathbb{R}. \tag{1}$$

Answer the following questions.

(i) Suppose that $u(x)$ is a solution of (1) and consider a new dependent variable $y(x) = xu(x)$. Derive an ordinary differential equation for y.

(ii) Find the general solution u of (1).

(iii) Find a particular solution u of (1) such that u is bounded on $\mathbb{R}$ and $u(0) = 1$.

5 (Selection problem)

For an arbitrary positive constant ε, let r_ε be a rectangular function defined by

$$r_\varepsilon(x) = \begin{cases} 0 & \text{if } |x| > \varepsilon, \\ \frac{1}{4\varepsilon} & \text{if } |x| = \varepsilon, \\ \frac{1}{2\varepsilon} & \text{if } |x| < \varepsilon. \end{cases}$$

For an arbitrary function f absolutely integrable on $\mathbb{R}$, let $\mathcal{F}[f]$ be a complex valued function defined by

$$\mathcal{F}[f](\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx, \quad \forall \xi \in \mathbb{R},$$

where i is the imaginary unit.

Answer the following questions.

(i) Compute $\mathcal{F}[r_\varepsilon](\xi)$ and express the result without using the imaginary unit.

(ii) For an arbitrary sequence $\{a_n\}_{n=-\infty}^{\infty}$ of real numbers, let s_ε be a step function defined by

$$s_\varepsilon(x) = \sum_{n=-\infty}^{\infty} a_n r_\varepsilon(x - 2n\varepsilon).$$

Calculate $\mathcal{F}[s_\varepsilon]$.

(iii) For an arbitrary continuous function f on $\mathbb{R}$, calculate $\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} r_\varepsilon(x) f(x)\, dx$.

6 (Selection problem)

Let h be a positive constant. For an arbitrary integer n, let sinc_n be defined by

$$\mathrm{sinc}_n(x) = \mathrm{sinc}\left(\frac{\pi}{h}(x - nh)\right), \quad \forall x \in \mathbb{R},$$

where

$$\mathrm{sinc}(x) = \begin{cases} \frac{\sin x}{x} & (x \neq 0), \\ \lim\limits_{x \to 0} \frac{\sin x}{x} & (x = 0). \end{cases}$$

Answer the following questions.

(i) For an arbitrary integer m, calculate $\mathrm{sinc}_n(mh)$.

(ii) Let f_h be a linear combination of sinc_n, i.e.,

$$f_h(x) = \sum_{n=-\infty}^{\infty} a_n \mathrm{sinc}_n(x), \quad \forall x \in \mathbb{R}$$

where $\{a_n\}_{n=-\infty}^{\infty}$ is a sequence of real numbers. For an arbitrary continuous function f on $\mathbb{R}$, find the coefficients $\{a_n\}_{n=-\infty}^{\infty}$ satisfying

$$f_h(mh) = f(mh), \quad \forall m \in \mathbb{Z}.$$

(iii) For a non-negative integer N, let f_h^N be a truncation of f_h defined by

$$f_h^N(x) = \sum_{n=-N}^{N} a_n \mathrm{sinc}_n(x), \quad \forall x \in \mathbb{R}.$$

Find an approximation of $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^4}\, dx$ by using f_h^N with $h = 2$, $N = 2$, and a suitable f. Here, the following fact may be used:

$$\int_{-\infty}^{\infty} \mathrm{sinc}_0(x)\, dx = h.$$