# Selection Examination for the Training Program

# Mathematics

**1** (THIS QUESTION MUST BE ANSWERED)

Let $f_n$ be a function on $(-1,1)$ defined by

$$f_n(x) = \sin\bigl(n \arcsin(x)\bigr),$$

where $n$ is an arbitrary non-negative integer.

Answer the following questions.

(i) Calculate $f_n'$, where $f_n'$ means the first derivative of $f_n$ with respect to $x$.

(ii) Prove that $f_n$ satisfies the following differential equation:

$$(1-x^2)f_n''(x) - xf_n'(x) + n^2 f_n(x) = 0, \quad \forall x \in (-1,1).$$

(iii) For any non-negative integer $m$, calculate the $m$-th order differential coefficient $f_n^{(m)}(0)$.

**2** (THIS QUESTION MUST BE ANSWERED)

Let $\mathbb{P}^3$ be a set of all real polynomials of degree less than or equal to three. Prove or disprove that the following $W$ is a linear subspace of $\mathbb{P}^3$. Moreover, if $W$ a linear subspace of $\mathbb{P}^3$ then calculate the basis of $W$.

(i) $W = \bigl\{f \in \mathbb{P}^3 \ ; \ f(0) = 0, \quad f(1) = 1\bigr\}$.

(ii) $W = \bigl\{f \in \mathbb{P}^3 \ ; \ f(0) = 0, \quad f'(0) = 0\bigr\}$.

(iii) $W = \bigl\{f \in \mathbb{P}^3 \ ; \ xf'(x) - 2f(x) = 0\bigr\}$.

$\boxed{3}$ (Selection problem)

Let $z_n$ be a sequence in $\mathbb{C}$ for which there exists a positive integer $N$ such that $\sup_{n \geq N} \frac{|z_{n+1}|}{|z_n|} < 1$. Let $\alpha_N = \sup_{n \geq N} \frac{|z_{n+1}|}{|z_n|}$.

Answer the following questions.

(i) Prove that

$$|z_n| \leq \alpha_N^{n-N} |z_N|, \quad \forall n \geq N.$$

(ii) Prove that

$$\sum_{n=N}^{\infty} |z_n| \leq \frac{|z_N|}{1 - \alpha_N}.$$

(iii) Let $S_N$ and $S$ be defined by

$$S_N = \sum_{n=1}^{N} (-1)^{n-1} \frac{(1+i)^{2n-1}}{(2n-1)!} \quad \text{and} \quad S = \lim_{N \to \infty} S_N,$$

respectively. Here, $i$ is an imaginary unit. Calculate a natural number $N$ for which $|S - S_{N-1}|$ becomes at least $10^{-2}$ or less.

$\boxed{4}$ (Selection problem) Let $\lambda$ be a positive constant. We consider the following forced vibration model:

$$\begin{cases} \dfrac{d^2u}{dt^2}(t) + \lambda^2 u(t) = \lambda^2 \cos(t), & \forall t > 0, \\ u(0) = \frac{du}{dt}(0) = 0. \end{cases}$$

Answer the following questions.

(i) Calculate the general solution of the following:

$$\frac{d^2v}{dt^2}(t) + \lambda^2 v(t) = 0, \quad \forall t > 0.$$

(ii) Solve $u$.

(iii) What is the condition on $\lambda$ so that the amplitude of $u$ growth as $t \to \infty$ ? Moreover, calculate the growth order of $|u|$ for $t$.

$\boxed{\mathbf{5}}$ (Selection problem)

Let $a$ and $b$ be real constants satisfying $-\infty < a < b < \infty$. $s_n$ is defined by

$$s_n(x) = \sqrt{\frac{2}{b-a}} \sin\left(\frac{x-a}{b-a} n\pi\right)$$

for arbitrary positive integer $n$.

Answer the following questions.

(i) Prove that $s_n$ satisfies the following ordinary differential equation:

$$s_n''(x) = -\left(\frac{n\pi}{b-a}\right)^2 s_n(x), \quad \forall x \in (a,b)$$

and boundary conditions:

$$s_n(a) = s_n(b) = 0$$

for arbitrary positive integer $n$.

(ii) Let $\alpha_n$ be a positive constants for arbitrary positive integer $n$. Solve the following initial value problem:

$$\begin{cases} \dfrac{df_n}{dt}(t) = -\alpha_n^2 f_n(t), & \forall t \in (0,\infty), \\ f_n(0) = 1. \end{cases}$$

(iii) Let $\beta_n$ be real constants satisfying $\sum_{n=1}^{\infty} n^2 |\beta_n| < \infty$. Solve the following initial-boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) = 0, & \forall (x,t) \in (a,b) \times (0,\infty), \\ u(a,t) = u(b,t) = 0, & \forall t \in (0,\infty), \\ u(x,0) = \displaystyle\sum_{n=1}^{\infty} \beta_n s_n(x), & \forall x \in (a,b). \end{cases}$$

$\boxed{\mathbf{6}}$ (Selection problem)

Let $P_n$ be a polynomial of degree $n$ defined by

$$P_n(x) = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n (x^2-1)^n$$

where $n$ is an arbitrary non-negative integer. These $P_n$'s satisfy

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} \dfrac{2}{2n+1}, & \text{if } m=n, \\ 0, & \text{if } m \neq n. \end{cases}$$

All roots of $P_n$ exist in $(-1,1)$ and these are simple.

Answer the following questions.

(i) For $f(x) = 5x^5 - 8x^3 + 2x^2$, calculate the polynomial quotient of $q$ and the polynomial remainder $r$ related to $P_3$ by

$$f(x) = P_3(x)q(x) + r(x).$$

(ii) Let $\{\xi_i\}_{i=1}^n$ be the roots of $P_n$. Let $\{\phi_i\}_{i=1}^n$ and $\{w_i\}_{i=1}^n$ be polynomials of degree $n-1$ whose integrals are defined by

$$\phi_i(x) = \prod_{\substack{j=1\\ i\neq j}}^{n} \frac{x-\xi_j}{\xi_i-\xi_j}, \quad w_i = \int_{-1}^{1} \phi_i(x)\,dx, \quad \forall i \in \{1,2,\ldots,n\},$$

respectively. For an arbitrary polynomial $f$ of degree less than or equal to $2n-1$, prove the following equation:

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} f(\xi_i)w_i.$$

(iii) Let $\{\xi_i\}_{i=1}^3$ be the roots of $P_3$ satisfying $-1 < \xi_1 < \xi_2 < \xi_3 < 1$. Then,

$$w_1 = \frac{5}{9}, \quad w_2 = \frac{8}{9}, \quad w_3 = \frac{5}{9}$$

are satisfied. Calculate $\log(2)$ up to four decimal places. You do not have to prove that the approximate value is correct up to four decimal.