Selection Examination for the Training Program

Mathematics

1 (THIS QUESTION MUST BE ANSWERED)

Let X and Y be vector spaces over $\mathbb{R}$. A mapping, usually shortened to map, f is defined by a correspondence relation from any element x in X to an element $f(x)$ in Y and the map is written by $f : X \to Y$. Choose all the maps from below that is **NOT** real linear map giving a reason why it is not linear map.

(i) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x + 1.$

(ii) $f : \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x_1, x_2) = \begin{pmatrix} x_1 + 2x_2 \\ -x_1 \end{pmatrix}.$

(iii) $f : C^1(\mathbb{R}) \to C^0(\mathbb{R}), \quad f(u) = \dfrac{du}{dx} + x^2 u$, where $C^m(\mathbb{R})$ is the set of all real functions with differentiability class C^m.

(iv) $f : C^0([0,1]) \to \mathbb{R}, \quad f(u) = \displaystyle\int_0^1 \big(u(x)\big)^2 \, dx.$

2 (THIS QUESTION MUST BE ANSWERED)

For a real function f, we consider the Newton method which is defined by the following recurrence relation

$$x_{n+1} = x_n - \left(\frac{df}{dx}(x_n) \right)^{-1} f(x_n),$$

where n is an arbitrary non-negative integer.

Answer the following questions.

(i) For $f(x) = \dfrac{1}{x^3} - a$, where a is a real constant, determine a recurrence relation by the Newton method.

(ii) Calculate $\dfrac{1}{\sqrt[3]{2}}$ to three decimal places. You do not have to prove that the approximate value is correct up to three decimal.

3 (Selection problem)

Let f be a complex function from $\mathbb{C}$ to $\mathbb{C}$ satisfying $f(z) = u(x,y) + iv(x,y)$, where u and v are real functions from $\mathbb{R}^2$ to $\mathbb{R}$, $z = x + iy \in \mathbb{C}$ and i is the imaginary unit. We define the complex derivative of f at the point z by

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}.$$

If the following functions $f = u + iv$ have the complex derivative for all z in $\mathbb{C}$, calculate $f'(z)$. Otherwise, calculate one of points z_0 at which the complex derivative $f'(z_0)$ does not exist.

(i) $u(x,y) = x$ and $v(x,y) = -y$.

(ii) $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$.

(iii) $u(x,y) = \frac{e^{-y}+e^{y}}{2} \cos x$ and $v(x,y) = \frac{e^{-y}-e^{y}}{2} \sin x$.

(iv) There exist C^0 functions ϕ and ψ satisfying the following condition:

$$\begin{pmatrix} \frac{\partial u}{\partial x}(x,y) & \frac{\partial u}{\partial y}(x,y) \\ \frac{\partial v}{\partial x}(x,y) & \frac{\partial v}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} \phi(x,y) & -\psi(x,y) \\ \psi(x,y) & \phi(x,y) \end{pmatrix}$$

for all (x,y) in $\mathbb{R}^2$.

4 (Selection problem)

Answer the following questions.

(i) Let A be a 2-by-2 matrix defined by

$$A = \frac{1}{2} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

Calculate an orthogonal matrix P and a diagonal matrix D satisfying $D = P^T A P$, where $\cdot^T$ means the transpose.

(ii) Solve the following ordinary differential equations

$$\begin{aligned}
\frac{du_1}{dt}(t) &= \frac{3}{2}u_1(t) + \frac{1}{2}u_2(t), \quad t > 0, \\
\frac{du_2}{dt}(t) &= \frac{1}{2}u_1(t) + \frac{3}{2}u_2(t), \quad t > 0, \\
u_1(0) &= c_1, \\
u_2(0) &= c_2,
\end{aligned}$$

where c_1 and c_2 are real constants.

$\boxed{5}$ (Selection problem)

Let a and b be real constants satisfying $-\infty < a < b < \infty$. c_n and s_n are defined by

$$c_n(x) = \sqrt{\frac{2}{b-a}} \cos\left(\frac{2x-b-a}{b-a}n\pi\right) \quad \text{and} \quad s_n(x) = \sqrt{\frac{2}{b-a}} \sin\left(\frac{2x-b-a}{b-a}n\pi\right)$$

for arbitrary positive integer n, respectively. Moreover, c_0 is defined by $c_0 = \frac{1}{\sqrt{b-a}}$. Let e_n be defined by

$$e_n(x) = \frac{1}{\sqrt{b-a}} e^{\frac{2x-b-a}{b-a}n\pi i}$$

for arbitrary integer n, where i is the imaginary unit.

Answer the following questions.

(i) Calculate the following integral

$$\int_a^b e_m(x)\overline{e_n(x)}\, dx$$

for arbitrary integer m and n, where $\overline{\cdot}$ means the complex conjugate.

(ii) Let α_n and β_n be real constants and let f be defined by

$$f(x) = \alpha_0 c_0 + \sum_{n=1}^{\infty}\big(\alpha_n c_n(x) + \beta_n s_n(x)\big).$$

Calculate the complex number γ_n defined by

$$\gamma_n = \int_a^b f(x)\overline{e_n(x)}\, dx$$

for arbitrary integer n.

6 (Selection problem)

For arbitrary real symmetric positive definite matrix A, there exists a real lower triangular matrix L such that $A = LL^T$ where $\cdot^T$ means the transpose. This procedure is called the Cholesky decomposition and L is called Cholesky factor of A.

Let A be a n-by-n real symmetric positive definite matrix defined by

$$A = \begin{pmatrix} A_{1,1} & A_{2,1} & \cdots & A_{n,1} \\ A_{2,1} & A_{2,2} & \cdots & A_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,1} & A_{n,2} & \cdots & A_{n,n} \end{pmatrix}.$$

One of algorithms of the Cholesky decomposition which uses same memory as A is given as follows.

for $j = 1$ to n
    $A_{j,j} := \sqrt{A_{j,j}}$
    for $i = j + 1$ to n
        $A_{i,j} := A_{i,j}/A_{j,j}$
    end for
    for $k = j + 1$ to n
        for $i = k$ to n
            $A_{i,k} := A_{i,k} - A_{i,j}A_{k,j}$
        end for
    end for
end for

Answer the following questions.

(i) For $n = 3$, calculate the Cholesky factor L of A.

(ii) Calculate the computational complexity of this algorithm, where the computational complexity is defined by the number of times of the four arithmetic operations and square root for the highest order of n.