Selection Examination for the Training Program

Mathematics

1 (THIS QUESTION MUST BE ANSWERED)

Let D be a subset of $\mathbb{R}^2$ defined by

$$D = \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, 2 \leq x^2 + y^2 \leq 4, \quad 0 < y \leq x \leq \sqrt{3}y \right\}.$$

Answer the following questions.

(i) For arbitrary $(x, y) \in D$, let $u(x, y)$ and $v(x, y)$ be defined by

$$u(x, y) = \frac{1}{2}(x^2 + y^2), \quad v(x, y) = \frac{x}{y}.$$

Calculate the following determinant:

$$\det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

(ii) Calculate the following double integral:

$$\iint_D \left(1 + \frac{2x}{y} \right) dxdy.$$

2 (THIS QUESTION MUST BE ANSWERED)

Let p and q be real numbers. Answer the following questions.

(i) Determine the rank of matrix A defined by

$$A = \begin{pmatrix} 1 & p & -1 & 2 & 1 \\ 2 & p^2 + 2p - 1 & 1 & 2 & q^2 \\ 1 & -2p^2 - 3p - 2 & -7 & 6 & -2q^2 + q + 2 \\ 0 & -p^2 + 1 & 3 & -2 & q^2 + q - 5 \end{pmatrix}.$$

(ii) Suppose the rank of A is equal to two. Determine a necessary and sufficient condition for the real numbers b_1, b_2, b_3, b_4 to be solved the following simultaneous linear equations with respect to $x_1, x_2, \ldots, x_5$:

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.$$

3 (Selection problem)

We consider the smooth curve C passed the point $(1, 1)$ on the xy-plane. Let X be an intersection point of a tangential line at A on this curve and x-axis. Here we assume the length of line segments of OX and AX coincide for arbitrary A. Express the equation of C in terms of x and y. Here, O is the origin of the coordinate xy-axes.

4 (Selection problem)

Let ξ be an arbitrary positive number. Calculate the integral

$$\int_{-\infty}^{\infty} \frac{e^{-i\xi x}}{1+x^2}\, dx,$$

where i is the imaginary unit.

5 (Selection problem)

Let $C^1_{\#}([-\pi, \pi])$ be a set of the functions of the class C^1 with periodic boundary conditions on the closed interval $[-\pi, \pi]$, namely,

$$C^1_{\#}([-\pi, \pi]) = \left\{ f \in C^1([-\pi, \pi]) \,\middle|\, f(-\pi) = f(\pi),\ f'(-\pi) = f'(\pi) \right\}.$$

If $f \in C^1_{\#}([-\pi, \pi])$ is not identically zero and $\int_{-\pi}^{\pi} f(x)\, dx = 0$, we define $A(f)$ and $B(f)$ by

$$A(f) = \int_{-\pi}^{\pi} |f(x)|^2\, dx, \quad \text{and} \quad B(f) = \int_{-\pi}^{\pi} |f'(x)|^2\, dx$$

respectively. Prove

$$\inf_{f \in C^1_{\#}([-\pi,\pi])} \frac{B(f)}{A(f)} > 0.$$

6 (Selection problem)

Let a, b, c, and d be real numbers and let T be a positive number. For given $u(0)$ and $v(0)$ in $\mathbb{R}$, we consider $u(t)$ and $v(t)$ satisfying the following initial values problem for ordinary differential equations:

$$\begin{cases} \dfrac{du}{dt} = au + bv, & 0 < t < T, \\ \dfrac{dv}{dt} = cu + dv, & 0 < t < T. \end{cases}$$

Let $\{t_n\}_{n=0}^N$ be the uniformly nodal points on $[0,T]$ satisfying $0 = t_0 < t_1 < \cdots < t_N = T$, and set $\Delta t_n = t_{n+1} - t_n$, $(0 \le n \le N-1)$.

Under the above assumption, we define u_n and v_n as the approximate values of $u(t_n)$ and $v(t_n)$, respectively, which satisfy the following difference equation:

$$\begin{cases} \dfrac{u_{n+1} - u_n}{\Delta t_n} = au_{n+1} + bv_n, \\ \dfrac{v_{n+1} - v_n}{\Delta t_n} = cu_{n+1} + dv_n. \end{cases}$$

Answer the following questions.

(i) Let A be a 2-by-2 matrix satisfying

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = A \begin{pmatrix} u_n \\ v_n \end{pmatrix}.$$

Express all the entries of A in terms of a, b, c, d, and Δt_n.

(ii) Such that the signed area of the triangle generated by three points $(0,0)$, (u_n, v_n), and (u_{n+1}, v_{n+1}) is constant independently on initial values and nodal points. Find sufficient and necessary conditions on a, b, c, and d.